THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR LINEARIZED KdV EQUATION

Barabash Tatyana O. , Eidel'man Samuil D. (Ukraine,Kiev)

1.Statement of the problem. We investigate the asymptotic behaviour (when $t \to \infty$) of solutions of boundary-value problems for equation

$$\partial_t u(t,x) = -\partial_x^3 u(t,x). \tag{1}$$

This is a linearized Korteweg-de-Vries equation. We study solutions of the equation (1) in the domain $E^2_{++} = \{(t,x): t>0, x>0\}$ natural from the physical point of view. Equation (1) is well-posed in the sense of Petrovskij and belongs to the class of q-hyperbolic equations (q=3), which were defined by Gindikin S.G. in [1]. The theory of mixed (boundary) problems for such equation is stated in important works of L.P.Valevich and S.G.Gindikin [2,3]. Here we consider boundary-value problems for equation (1) of the following type

$$u(t,x)\,\Big|_{+0} = 0, x \in R^1_+ = \{x, x>0\} \tag{2}$$

$$B(\partial_x)u(t,x)\Big|_{x=+0} = g(t), t \in E^1_+ = \{t: t>0\} \tag{3}$$

where $B(\lambda) = \sum_{k=0}^{r} b_k \lambda^k$ - polynomial with real coefficients.

We find condition on the poiinomial $B(\lambda)$, which guarantees stabilization of bounded solutions $u(t,x)$ of the problem (1)-(3), constructed by arbitrary stabilizing boundary functions $g(t)$.

2 **Dirichlet problem.** In our further analysis the propreties of the Poisson's kernel of the problem (1)-(3) play an essential role, particularly, their behaviour at large time intervals. From our point of view these properties are very interesting for themselves. First, let us consider the Dirichlet problem ($B(\lambda)\equiv 1$). As a result of this consideration, information important for all further arguments will be obtained.

Theorem1. The Poisson kernel $G_0(t,x)$ of the Dirichlet problem for equation (1) is defined as

$$G_0(t,x) = xt^{-4/3}\mathrm{Ai}(xt^{-1/3}), \tag{4}$$

where $\mathrm{Ai}(z)$ -is the Airy function ([3],10.4.32).

Proof. Here we apply the method similar to that of surface potentials. According to this method, at first the fundamental solution $Z(t,x)$ of the Cauchy problem for equation (1) must be find by means of usual Fourier-transform with respect to space coordinate x, we receive, Using formula (10.4.32 [3]), it is possible to express (5) in terms of the Airy function:

$$Z(t,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\{i\sigma x+i\sigma^3 t\}d\sigma \tag{5}$$

$$Z(t,x)=t^{-1/3}Ai(xt^{-1/3}). \tag{6}$$

Let us recall, that for $|\arg z|<\pi$ and $|z|\to\infty$ the following asymptotic series presents the Airy function (10.4.59,[3]):

$$Ai(z)\approx\frac{1}{2}\pi^{-1/2}z^{-1/4}\exp\{-\xi\}\sum_{k=0}^{\infty}(-1)^k c_k\xi^{-k},\xi=\frac{2}{3}z^{3/2} \tag{7}$$

Solution of the Dirichlet problem for equation (1), which saticfies the zeroth initial conditions, is searched for in the form of binary layer potential:

$$u_1(t,x)=\int_0^t\partial_x^2 Z(t-\tau,x)\psi(\tau)d\tau, \tag{8}$$

where density-function $\psi(\tau)$ need to be defined. Using the differential equation for the Airy function $w''-zw=0$ (10.4.1,[3]) one can present $u_1(t,x)$ as:

$$u_1(t,x)=\int_0^t x(t-\tau)^{-4/3}Ai(x(t-\tau)^{-1/3})\psi(\tau)d\tau=\int_{xt^{-1/3}}^{\infty}3Ai(z)\psi(t-(x/z)^3)dz. \tag{9}$$

Let $x\to0$ in the equality (9). Then one can receive, that:

$$u(t,0)=\psi(t)\int_0^{\infty}3Ai(z)dz.$$

Applying the asymptotic series (10.4.82[3]) for the Airy function, from which follows, that

$\int_0^{\infty}Ai(\beta)d\beta=\frac{1}{3}$. That means, that $\psi(t)=g(t)$.

Then we ultimately receive solution of the Dirichlet problem represented by formula:

$$u(t,x)=\int_0^{\tau}G_0(t,x)g(t-\tau)d\tau, \tag{10}$$

where $G_0(t,x)=xt^{-4/3}Ai(xt^{-1/3})$ the Poisson kernel for the Dirichlet problem.

3.**General boundary problem.** Now let us pass to the investigation of the boundary-value problems (1)-(3) in the most general case. Applying the Laplace-transform with respect to time coordinate t to equation (1), one can receive:

$$\frac{d^3\hat{u}}{dx^3}+p\hat{u}=0,B(\frac{d}{dx})\hat{u}(p,x)\ |_{x=0}=\hat{g}(p),\hat{u}(p,x)\to0,x\to\infty, \tag{11}$$

where functions $\hat{g}(p)$ and $\hat{u}(p,x)$ - Laplace-images of functions g(t) and u(t,x) correspondingly. These functions are considered for x>0 and Rep>0. Characteric equation, corresponding to (10) looks like

$$\lambda^3 + p = 0, p = |p|\exp\{i\varphi\}, \tag{12}$$

and has the following solutions:

$$\lambda_j(p) = |p|^{1/3}\exp\{i((\varphi+\pi)/3 + 2\pi(j-1)/3)\}, j = 1,2,3. \tag{13}$$

For {p:Rep>0} these solutions are situated at the appropriate sectors,presented at Fig.1.

Only one root $\lambda_2(p)$ among these three ones, has negative real part, and therefore it defines the solutions of problem (11)-(12) by formula:

$$\hat{u}(p,x) = \hat{g}(p)B(\lambda_2(p))^{-1}\exp\{\lambda_2(p)x\}.$$

Thus, we have received the representation for bounded solutions u(t,x) of the problem (1)-(3) as

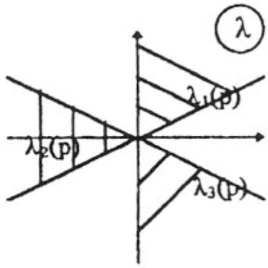

Fig.1.

$$u(t,x) = \int_0^t G(t-\tau,x)g(\tau)d\tau, \tag{14}$$

where

$$G(t,x) = (2\pi i)^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\{pt + \lambda_2(p)x\}(B(\lambda_2(p)))^{-1}dp \quad - \tag{15$_1$}$$

the Poisson kernel of the problem (1)-(3), and γ - suitable, generally speaking, large enough positive constant.

The analysis of formula (15) leads to the statement of the following main condition.

Condition α. All zeros of polynomial $B(\lambda)$ are situated at the sector $\Lambda = \{\lambda \in C: \arg\lambda \in (-5\pi/6, +5\pi/6\}$.

This condition is equivalent to the requirement that all zeros of the algebraic function $F(p)=B(\lambda_2(p))$ are situated at the half-plane Rep<0 of the complex p-plane.

4.Poisson kernels of general boundary -value problems. Let us pass to the investigation of Poisson kernels G(t,x) of boundary-value problems (1)-(3).

Theorem 2. Let the condition α be held. Then

1) for $t \in (0,1], x > 0$, the following estimation is valid

$$|G(t,x)| \le Ct\exp\{-c(xt^{-1/3})^{3/2}\}; \tag{15$_2$}$$

2)the following representation takes place:

$$G(t,x) = G_0(t,x)B(0)^{-1} + G_1(t,x), \tag{16}$$
where for $t \geq 1$ for the function $G_1(t,x)$ the following estimation is valid:

$$|G_1(t,x)| \leq Ct^{-4/3}\exp\{(-c|x|t^{-1/3})^{3/2}\} \tag{17}$$

3)integral $\int_0^\infty |G(t,x)|dt$ is uniformly bounded with respect to x.

<u>Proof.</u> Let us concisely set forth the proof of the statement 2). Statement 1) is prooved by the help of analogous concept then, but it is essentially more simple. Statement 3) immediately follows from 1) and 2).

Let us use the identity:

$$B(\lambda_2(p))^{-1} = B(0)^{-1} + (B(0) - B(\lambda_2(p)))(B(0)B(\lambda_2(p)))^{-1}.$$

By means of it, Poisson kernel G(t,x) can be written in the form of (16), where

$$G_1(t,x) = -(2\pi i)^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\{pt + \lambda_2(p)x\}\sum_{v=1}^{r} b_v(\lambda_2(p))^v(B(0)B(\lambda_2(p)))^{-1}dp,$$

where γ - arbitrary positive number. Sebsequently introducing new integral variables $q = pt, \hat{q} = \hat{x}^{-3/2}q, \hat{x} = xt^{-1/3}, \hat{x} \geq 1$, and using systematically the Cauchy theorem from the theory of analytical functions, we receive equality

$$G_1(t,x) = \frac{-\hat{x}^{3/2}}{2\pi it}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\{(\hat{q} + \lambda_2(\hat{q}))\hat{x}^{3/2}\}\sum_{v=1}^{r}\frac{b_v\lambda_2(\hat{q})^v(\hat{x}^{1/2}t^{-1/3})^v}{B(0)B(\lambda_2(\hat{q}))\hat{x}^{1/2}t^{-1/3}}d\hat{q}. \tag{18}$$

In the last integral let us pass from integration along straight line, parallel to imaginary axis ($\gamma-i\infty$, $\gamma+i\infty$) to integration along two rays that comes from the point $\gamma+i0$ and makes with imaginary axis angle φ_0 under $\operatorname{Im}\hat{q} > 0$ and $-\varphi_0$ under $\operatorname{Im}\hat{q} < 0$, φ_0 -rather small positive number. Such transition is possible because of the analyticity of integrand function due to the condition α, which is guaranteed in the region

$$\Omega = \{\hat{q}: \hat{q} = \gamma + iv\exp\{i\,\mathrm{sign}\,v\varphi\}, v \in (-\infty,\infty), 0 \leq \varphi \leq \varphi_0\}.$$

This region is contained between the straight line $\operatorname{Re}\hat{q} = \gamma$ and rays

$J_\gamma = \{\hat{q}: \hat{q} = \gamma + iv\exp(i\,\mathrm{sign}\,v\varphi_0), v \in (-\infty,\infty)\}$ (Fig.2)

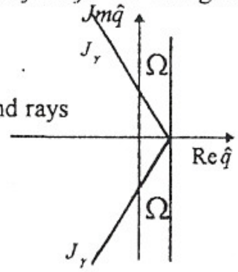

Fig.2.

In order to estimate $G_1(t,x)$, which is defined by integral with J_γ, it is necessary to carry out

$$\hat{q} = \gamma - |v|\sin\varphi_0 + iv\cos\varphi_0, \hat{q} \in J_\gamma,$$

the following calculations: $|\hat{q}|^2 = \gamma^2 + v^2 - 2\gamma v\sin\varphi_0 \geq (\gamma^2 + v^2)(1 - \sin\varphi_0),$ (19_1)

$$|\hat{q}|^2 \leq \gamma^2 + v^2,$$

$$(\sqrt[3]{\gamma} + \sqrt[3]{|v|})^6 \leq 2^6 \max(\sqrt[3]{\gamma}, \sqrt[3]{|v|})^6 \leq 64(\gamma^2 + v^2),$$

$$\sqrt[3]{|\hat{q}|} \geq \frac{1}{64}(1 - \sin\varphi_0)^{1/6}(\sqrt[3]{\gamma} + \sqrt[3]{|v|}),$$

$$\hat{q} = |\hat{q}|\exp\{i\psi_0\}, |\psi_0| = (\pi/2) + \varphi_0.$$

$$\operatorname{Re}\lambda_2(\hat{q}) = -\sqrt[3]{|\hat{q}|}\cos\psi_0 \leq -\frac{1}{64}(1 - \sin\varphi_0)^{1/6}\cos(\frac{\pi}{6} + \frac{\varphi_0}{3})(\sqrt[3]{\gamma} + \sqrt[3]{|v|}) =$$

$$= -\delta_1(\sqrt[3]{\gamma} + \sqrt[3]{|v|}), \delta_1 > 0.$$

$$(19_2)$$

$$\left|\exp\{(\hat{q} + \lambda_2(\hat{q}))\hat{x}^{3/2}\}\right| \leq \exp\left\{\gamma - |v|\sin\varphi_0 - \delta_1(\sqrt[3]{\gamma} + |v|^{1/3})\hat{x}^{3/2}\right\} =$$

$$= \exp\left\{[(\gamma - \delta_1\gamma^{1/3}) - |v|\sin\varphi_0 - \delta_1|v|^{1/3}]\hat{x}^{3/2}\right\},$$

where $-\delta_2 = \gamma - \delta_1\gamma^{1/3}$ can be made negative thanks to appropriate choice of a rather small γ.

Condition α results in the existence of positive constant δ_0 such; that

$$\left|B(0)B(\lambda_2(\hat{q})\hat{x}^{1/2}t^{-1/3})\right| \geq \delta_0. \qquad (19_3)$$

Equality (18), inequality $(19_{1,2,3})$ result in the following:

$$|G_1(t,x)| \leq$$

$$\leq C_1\frac{\hat{x}^{3/2}}{t}\int_{J_\gamma}\left|\exp((\hat{q} + \lambda_2(\hat{q}))\hat{x}^{3/2})\right|\sum_{v=1}^{r} b_v(\lambda_2(\hat{q})\frac{\hat{x}^{1/2}}{t^{1/3}})(B(0)B(\lambda_2(\hat{q})\frac{\hat{x}^{1/2}}{t^{1/3}}))^{-1}d\hat{q}$$

$$\leq C_2\delta_0^{-1}\frac{\hat{x}^{3/2}}{t}\exp\{-\delta_2\hat{x}^{3/2}\}\int_{J_\gamma}\exp\left\{(-|v|\sin\varphi_0 - \delta_1|v|^{1/3})\hat{x}^{3/2}\right\}\sum_{v=1}^{r}(\gamma^2 + v^2)^{v/6}\frac{\hat{x}^{v/2}}{t^{v/3}}dv \leq$$

$$\leq C_3\frac{\hat{x}^{3/2+r/2}}{t^{4/3}}\exp\{-\delta_2\hat{x}^{3/2}\} \leq C_4(\varepsilon)t^{-4/3}\exp\{-(\delta_2 - \varepsilon)\hat{x}^{3/2}\} =$$

$$= C_4(\varepsilon)t^{-4/3}\exp\{-(\delta_2 - \varepsilon)(xt^{-1/3})^{3/2}\}. (20)$$

Thus, we have prooved property 2) for the Poisson kernel $G(t,x)$.

5.Stabilization. Now we pass to the proof of the theorem on stabilization.

Definition. Continuous along $[0,\infty)$ function $g(t)$ is called stabilizing, if there exists finite limit g_0 of the function $g(t)$ under $t\to\infty$.

Theorem 3.Let the condition α be fulfilled. Then solutions u(t,x) of the problem (1)-(3), being represented by the Poisson integral (14) and being constructed by any stabilizing boundary function g(t), is stabilizing uniformly in any segment [0,A] to the function

$$u_0(x) = g_0 \int_0^\infty G(t,x)dt.$$

Proof.Let us define function $v(t,x)=u(t,x)-u_0(x)$. Let us represent v (t,x) as the following:

$$v(t,x) = \int_0^t G(t-\tau,x)g(\tau)d\tau - \int_0^\infty G(t-\tau,x)g_0 d\tau = \int_0^t G(t-\tau,x)[g(\tau)-g_0]d\tau -$$

$$-\int_t^\infty G(t-\tau,x)g_0 d\tau = J_1(t,x) - J_2(t,x). \tag{21}$$

Let us consider the following function:

$$J_1(t,x) = \int_0^T G(t-\tau,x)(g(\tau)-g_0)d\tau + \int_T^t G(t-\tau,x)(g(\tau)-g_0)d\tau. \tag{22}$$

As $g(\tau)\to g_0$ under $\tau\to\infty$, then for any positive ε there exists such constant $T(\varepsilon)$, that for $\forall\tau > T(\varepsilon): |g(\tau)-g^0| < \varepsilon$. Let us consider, that $t > T(\varepsilon)+1$.

Then under $T=T(\varepsilon)$:

$$\left|\int_{T(\varepsilon)}^t G(t-\tau,x)(g(\tau)-g_0)d\tau\right| \le \varepsilon \int_{T(\varepsilon)}^t |G(t-\tau,x)|d\tau \le \varepsilon\int_0^\infty |G(\tau,x)|d\tau \le C\varepsilon \tag{23_1}$$

$$\left|\int_0^{T(\varepsilon)} G(t-\tau,x)(g(\tau)-g_0)d\tau\right| \le 2M \int_0^{T(\varepsilon)} |G(t-\tau,x|d\tau = 2M \int_{t-T(\varepsilon)}^t |G(\tau,x)|d\tau \tag{23_2}$$

$M=\sup|g(t)|$.

From the representation (16), condition α, evaluations $Ai(xt^{-1/3})$ and (17) the estimation follows (under $t\ge 1$, $x \le A$):

$$|G(t,x)| \le C(1+|x|)t^{-4/3} \le C(1+A)t^{-4/3}. \tag{23_3}$$

Expressions (23_2),(23_3) result in:

$$\int_0^{T(\varepsilon)} G(t-\tau,x)\big(g(\tau)-g_0\big)d\tau \le 6MC(1+A)\left[\big(t-T(\varepsilon)\big)^{\frac{-1}{3}} - t^{\frac{-1}{3}}\right] \tag{23_4}$$

To estimate $J_2(t,x)$ let us use evaluation (23_3):

$$|J_2(t,x)| = \left|g_0\int_t^\infty G(\tau,x)d\tau\right| \le |g_0|C(1+A)\int_t^\infty \tau^{-4/3}d\tau = 3C|g_0|(1+A)t^{-1/3}. \tag{23_5}$$

The statement of the theorem 3 follows from (23_1),(23_4),(23_5) .

The resarch was partially supported by grants of ISF and INTAS-94.

References

1.Gindikin S.G. Differential operators with real characteristics of high order//Dokl.Academy of Science of the USSR (1972),204,N5,p.1037-1040 (in Russian).
2.Volevich L.R., Gindikin S.G. Mixed problem for (2b+1)- hyperbolic equations//Trudi of Moscow Mathem. Society (1981),43,p.197-259 (in Russian).
3.Handbook of Mathematical Functions with Formulas, Graphs and Methenatical Tables. Edited by Melto Abramovitz and Irene A Stigun, National Boureau of Standards Applied Mathematics Series -55.